2020 B

Week 1 (Jan 11)

- Review of 1D integral
- Double integral over a rectangle

- Review of 1D integral.

Let f be a bounded function on $[a, b]$.
A partition P on $[a,b]$ is a collection of points $x_0, x_1, .. x_n$ satisfying

$$x_0 = a < x_1 < x_2 < \cdots < x_n = b.$$

Its norm is

$$\|P\| = \max\{\Delta x_1, \cdots, \Delta x_n\}, \quad \Delta x_j = x_j - x_{j-1}.$$

e.g. P on $[0, 1]$: $0 = x_0 < \frac{1}{3} < \frac{2}{3} < \frac{3}{4} < 1 = x_4$.

$\Delta x_1 = 1/3 - 0 = 1/3$, $\Delta x_2 = 2/3 - 1/3 = 1/3$, $\Delta x_3 = 3/4 - 2/3 = \frac{1}{12}$

$\Delta x_4 = 1 - 3/4 = 1/4$.

$\|P\| = \max\{1/3, 1/3, 1/12, 1/4\} = 1/3$.

The Riemann sum of f

$$S(f,P) = \sum_{j=1}^{n} f(z_j)\Delta x_j,$$ where $z_j \in [x_{j-1}, x_j]$ is a tag points.

The Riemann sum depends on P and the choice of tag points.

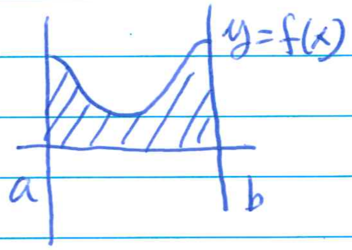
<u>Definition</u> A real number I is called the integral of f over $[a,b]$ if it satisfies the following property:
For each $\varepsilon > 0$, there exists some $\delta > 0$ such that

$$|S(f,P) - I| < \varepsilon \quad \text{whenever } P \text{ has norm } \|P\| < \delta.$$

Roughly speaking, whenever $\|P\|$ is sufficiently small,

$S(f,P)$ is close to I.

- When $f \geq 0$, I is the area of the region below the graph of $y=f(x)$ over $[a,b]$.

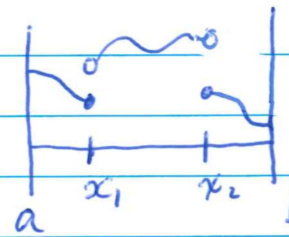


- Notation for I: $\displaystyle\int_a^b f(x)\,dx$, $\displaystyle\int_a^b f$, etc

- Functions whose integrals exist are called integrable fcns.
  ~ all continuous functions are integrable,
  ~ all piecewise functions are integrable,
  ~ there are non-integrable functions.

$$g(x) = \begin{cases} 1, & x \text{ rational} \\ 0, & x \text{ irrational} \end{cases}$$

g is not integrable on any $[a,b]$.



a piecewise conti fcn discont at x_1, x_2.

The evaluation of the integral depends on

<u>Fundamental theorem of Calculus</u> Let f be a continuous function on $[a,b]$ and F its primitive function. Then

$$\int_a^b f(x)\,dx = F(b) - F(a) \quad \left(= F(x)\Big|_a^b\right)$$

F is a primitive function of f if $F' = f$.

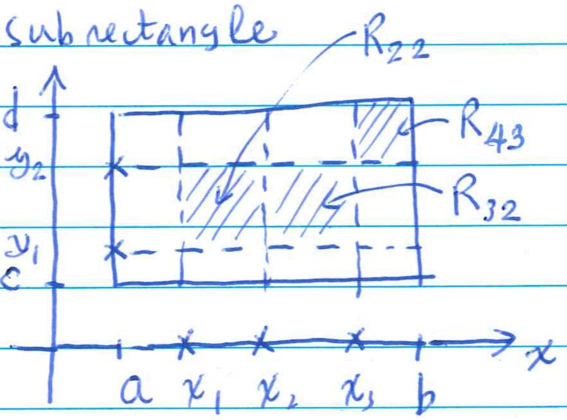
- Double Integral over a Rectangle

$$R = [a,b] \times [c,d]$$
$$= \{(x,y) : a \le x \le b,\ c \le y \le d\}.$$

Let P be a partition of R, i.e,

$$a = x_0 < x_1 < \dots < x_n = b$$
$$c = y_0 < y_1 < \dots < y_m = d$$

$R_{jk} = [x_{j-1}, x_j] \times [y_{k-1}, y_k]$ subrectangle



Let f be a bounded function on R. The Riemann sum is

$$S(f,P) = \sum_{j,k} f(P_{jk}) |R_{jk}|$$

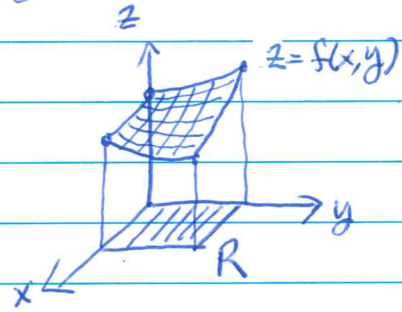
where $P_{jk} \in R_{jk}$ a tag point and $|R_{jk}| = \Delta x_j \Delta y_k$. The Riemann sum depends on the partition P and the choice of tag points.

A number I is called the integral of f over R if it satisfies, for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|S(f,P) - I| < \varepsilon \quad \text{whenever } \|P\| < \delta.$$

The norm of P, $\|P\| = \max\{\Delta x_1, \dots \Delta x_n, \Delta y_1, \dots \Delta y_m\}$.

- When $f \geq 0$ on R, I is the volume of the region bounded above by the graph of $z = f(x,y)$ over R.



- Notation for I:

$$\iint_R f(x,y)\,dA, \quad \iint_R f, \quad \text{etc}$$

- Just like 1-D case, there are non-integrable functions. On the other hand, all continuous functions are integrable.

Evaluation of double integral depends on

<u>Fubini's theorem</u> Let f be a continuous function on R.

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

$$= \int_c^d \int_a^b f(x,y)\,dx\,dy.$$

e.g. Evaluate $\iint_R (100-6x^2y)\,dA$, $R=[0,2]\times[-1,1]$.

By Fubini's theorem,

$$\iint_R (100-6x^2y)\,dA = \int_{-1}^{1}\int_0^2 (100-6x^2y)\,dx\,dy$$

$$= \int_{-1}^{1} \left(100x - 6\frac{x^3}{3}y\right)\Big|_{x=0}^{x=2} dy$$

$$= \int_{-1}^{1} (200 - 16y)\, dy$$

$$= (200y - 8y^2)\Big|_{-1}^{1}$$

$$= 400.$$

Alternatively,

$$\iint_R (100 - 6x^2y)\, dA = \int_0^2 \int_{-1}^{1} (100 - 6x^2y)\, dy\, dx$$

$$= \int_0^2 \left(100y - 6x^2 \frac{y^2}{2}\right)\Big|_{y=-1}^{y=1}$$

$$= \int_0^2 100 - 3x^2 - (100 \times -1 - 3x^2(-1)^2)\, dx$$

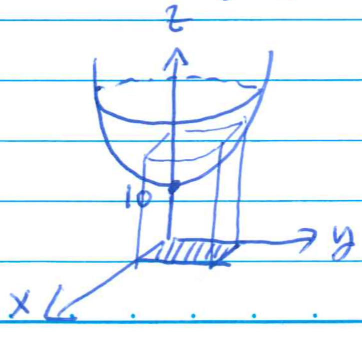
$$= \int_0^2 200\, dx$$

$$= 400,$$ the same result.

e.g. Find the volume of the region bounded above by the elliptical paraboloid $z = 10 + x^2 + 3y^2$ and below by $R: 0 \leq x \leq 1,\ 0 \leq y \leq 2$.

the volume ~~of~~ this region is $\iint_R (10 + x^2 + 3y^2)\, dA$, so is

$$\int_0^1 \int_0^2 (10 + x^2 + 3y^2)\, dy\, dx = \int_0^1 (10y + x^2y + y^3)\Big|_{y=0}^{y=2} dx$$



$$= \int_0^1 (20 + 2x^2 + 8)\, dx$$

$$= \frac{86}{3} \quad \#$$